10 LISREL with mean structures

10.1 The extended LISREL model

In Chapter 1 the LISREL model was defined by (1.1), (1.2), and (1.3), in which all random variables were assumed to have zero means. This assumption will now be relaxed and the model will be extended to include four new parameter matrices in addition to the previous eight. These new parameter matrices contain intercept term in the relationships and mean values of the latent variables.

The LISREL model is now defined by the following three equations corresponding to (1.1), (1.2), and (1.3), respectively:

$$\boldsymbol{\eta} = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta} , \qquad (10.1)$$

$$\mathbf{y} = \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\epsilon} , \qquad (10.2)$$

$$\mathbf{x} = \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta} , \qquad (10.3)$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\tau}_y$, and $\boldsymbol{\tau}_x$ are vectors of constant intercept terms. As before, we assume that $\boldsymbol{\zeta}$ is uncorrelated with $\boldsymbol{\xi}$, $\boldsymbol{\epsilon}$ is uncorrelated with $\boldsymbol{\eta}$ and that $\boldsymbol{\delta}$ is uncorrelated with $\boldsymbol{\xi}$. We also assume, as before, that $E(\boldsymbol{\zeta}) = \mathbf{0}$, $E(\boldsymbol{\epsilon}) = \mathbf{0}$, and $E(\boldsymbol{\delta}) = \mathbf{0}$, but it is not assumed that $E(\boldsymbol{\xi})$ and $E(\boldsymbol{\eta})$ are zero (E is the expected value operator). The mean of $\boldsymbol{\xi}$, $E(\boldsymbol{\xi})$, will be a parameter denoted by $\boldsymbol{\kappa}$. The mean of $\boldsymbol{\eta}$, $E(\boldsymbol{\eta})$, is obtained by taking the expectation of (10.1):

$$E(\boldsymbol{\eta}) = (\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa}) . \qquad (10.4)$$

By taking the expectations of (10.2) and (10.3), we find the mean vectors of the observed variables to be

$$\boldsymbol{\mu}_y = \boldsymbol{\tau}_y + \boldsymbol{\Lambda}_y(\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\kappa}) , \qquad (10.5)$$

$$\mu_x = \tau_x + \Lambda_x \kappa \ . \tag{10.6}$$

In general, in a single population, all the mean parameters τ_y, τ_x, α, and κ will not be identified without further conditions imposed. However, in simultaneous analysis of data from several groups, simple conditions (see Jöreskog & Sörbom, 1985) can be imposed to make all the mean parameters identified.

The LISREL model with mean structures introduces four new parameter matrices (actually vectors): τ_y, τ_x, α, and κ. These parameter matrices can be referred to in the same way as the other parameter matrices in LISREL. The LISREL notation and default forms for these are shown in Table 10.1. Each of these parameter matrices is fixed at zero by default. *They will be included in the model as soon as they are explicitly mentioned on the* MO *command.* They can be declared either fixed (FI), free (FR), or invariant (IN); or with the same pattern (SP), same starting values (SS), or both (PS) as in the previous group; see Section 9.2.

The fit function for the extended LISREL model with mean parameters is defined as

$$F = \sum_{g=1}^{G} \frac{N_g}{N} F_g \ , \tag{10.7}$$

where

$$F_g = (\mathrm{s}^{(g)} - \boldsymbol{\sigma}^{(g)})' \mathbf{W}_{(g)}^{-1} (\mathrm{s}^{(g)} - \boldsymbol{\sigma}^{(g)}) + (\bar{\mathrm{z}}^{(g)} - \boldsymbol{\mu}^{(g)})' \mathbf{V}_{(g)}^{-1} (\bar{\mathrm{z}}^{(g)} - \boldsymbol{\mu}^{(g)}) \tag{10.8}$$

Table 10.1 Additional Parameter Matrices in LISREL

Name	*Math Symbol*	*Order*	*LISREL Name*	*Possible Modes†*
TAU-Y	τ_y	NY × 1	TY	FI,FR,IN,PS,SP,SS
TAU-X	τ_x	NX × 1	TX	FI,FR,IN,PS,SP,SS
ALPHA	α	NE × 1	AL	FI,FR,IN,PS,SP,SS
KAPPA	κ	NK × 1	KA	FI,FR,IN,PS,SP,SS

† *Since these are vectors, Form is not relevant, only Mode is*

and $\boldsymbol{\mu}^{(g)} = (\boldsymbol{\mu}_y^{(g)}, \boldsymbol{\mu}_x^{(g)})'$.

The first term in (10.8) is the same as (1.15). The second term involves the sample mean vector $\bar{\mathbf{z}}^{(g)}$, the population mean vector $\boldsymbol{\mu}^{(g)}$, a function of parameters by (10.5) and (10.6), and the weight matrix $\mathbf{V}_{(g)}$ defined as:

$$\mathbf{V}_{(g)} = \mathbf{S}^{(g)} \qquad \text{for ULS, GLS, WLS, DWLS}$$

$$\mathbf{V}_{(g)} = \hat{\boldsymbol{\Sigma}}^{(g)} \qquad \text{for ML}$$

It should be noted that if $\boldsymbol{\tau}_y$, $\boldsymbol{\tau}_x$, $\boldsymbol{\alpha}$, and $\boldsymbol{\kappa}$ are all default, the second term in (10.8) is a constant, independent of parameters, in which case the problem reduces to the one described in Chapter 9 and no mean structures are needed.

If the observed variables have a multivariate normal distribution the ML case defined above yields maximum likelihood estimates in the sense of maximizing the multinormal likelihood function. Under the same assumption, ML and GLS give asymptotically efficient estimators. The fit function (10.7) may be justified under the more general assumption that $\bar{\mathbf{z}}^{(g)}$ and $\mathbf{S}^{(g)}$ are asymptotically uncorrelated. This holds, in particular, if the observed variables have no skewness. The case of completely non-normal distributions is dealt with in Section 7.4.

10.2 Estimation of factor means

Although the mean of a latent variable is undefined (not identified) in a single group, group differences in the means of latent variables can be estimated if the latent variables are on the same scale in all groups. Sörbom (1974) extended the classical model of factorial invariance, so that factor means could also be estimated. Sörbom's model is a LISREL Submodel 1 with mean structures defined as

$$\mathbf{x}^{(g)} = \boldsymbol{\tau}_x + \boldsymbol{\Lambda}_x \boldsymbol{\xi}^{(g)} + \boldsymbol{\delta}^{(g)}, \qquad g = 1, 2, \ldots, G\,,$$

with the mean of $\boldsymbol{\xi}^{(g)} = \boldsymbol{\kappa}^{(g)}$. To define the origins and the units of measurement of the ξ-factors, one can set $\boldsymbol{\kappa}^{(G)} = \mathbf{0}$ and fix one non-zero value in each column of $\boldsymbol{\Lambda}_x$. The parameters to be estimated are:

$\tau_x, \Lambda_x,$	assumed to be invariant over groups
$\kappa^{(1)}, \kappa^{(2)}, \ldots, \kappa^{(G-1)},$	mean vectors of ξ
$\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(G)},$	covariance matrices of ξ
$\Theta_\delta^{(1)}, \Theta_\delta^{(2)}, \ldots, \Theta_\delta^{(G)},$	error covariance matrices

The error covariance matrices Θ_δ may be postulated to be invariant over groups, if desired. In most cases these matrices are diagonal.

Example 10.1: Nine psychological variables with factor means

Source: Holzinger & Swineford (1939).

Nine variables were selected to measure three latent factors: Space, Verbal, and Memory. The groups consist of eighth-grade children from two schools in Chicago: the Pasteur and the Grant-White schools. The children from each school were divided into two groups according to whether they scored above or below the median on a speeded addition test. Thus the groups are:

1.	*Pasteur Low (N_1 = 77)*	*3.*	*Grant-White Low (N_3 = 74)*
2.	*Pasteur High (N_2 = 79)*	*4.*	*Grant-White High (N_4 = 71)*

The variables, correlations, standard deviations, and means are given in Tables 10.2 and 10.3. The standard deviations and means have been scaled so that the weighted estimate (see Section 9.3) of the within groups covariance matrix is a correlation matrix.

The nine labels are stored in the file EX101.LAB. All the remaining data are stored in the file EX101.DAT in the order: correlations for group 1, standard deviations for group 1, means for group 1, correlations for group 2, standard deviations for group 2, etc. In both files we use FORTRAN formats to record the data. Each group of data begins with a format line. After the last right parenthesis in the format, one can write any text. This can be used to identify the particular part of the data for which the format is intended. After the format follows the data.

The model used here is that outlined previously, with Λ_x and Θ_δ invariant over groups. The command file is shown below. The first line for each group is a title line. The DA command for the first group specifies

Table 10.2 Nine Psychological Variables: Correlations

Pasteur	*Group 1 above diagonal; Group 2 below diagonal*								
Test	1	2	3	4	5	6	7	8	9
Visual Perception	—	.32	.48	.28	.26	.40	.42	.12	.23
Cubes	.24	—	.33	.01	.01	.26	.32	.05	−.04
Paper Form Board	.23	.22	—	.06	.01	.10	.22	.03	.01
General Information	.32	.05	.23	—	.75	.60	.15	−.08	−.05
Sentence Completion	.35	.23	.18	.68	—	.63	.07	.06	.10
Word Classification	.36	.10	.11	.59	.66	—	.36	.19	.24
Figure Recognition	.22	.01	−.07	.09	.11	.12	—	.29	.19
Object-Number	−.02	−.01	−.13	.05	.08	.03	.19	—	.38
Number-Figure	.09	−.14	−.06	.16	.02	.12	.15	.29	—
Grant-White	*Group 3 above diagonal; Group 4 below diagonal*								
Test	1	2	3	4	5	6	7	8	9
Visual Perception	—	.34	.41	.38	.40	.42	.35	.16	.35
Cubes	.32	—	.21	.32	.16	.13	.27	.01	.27
Paper Form Board	.34	.18	—	.31	.24	.35	.30	.09	.09
General Information	.31	.24	.31	—	.69	.55	.17	.31	.34
Sentence Completion	.22	.16	.29	.62	—	.65	.20	.30	.27
Word Classification	.27	.20	.32	.57	.61	—	.31	.34	.27
Figure Recognition	.48	.31	.32	.18	.20	.29	—	.31	.38
Object-Number	.20	.01	.15	.06	.19	.15	.36	—	.38
Number-Figure	.42	.28	.40	.11	.07	.18	.35	.44	—

the number of groups: NG=4. The next line actually contains three commands, separated by semicolons. These commands tell the program to read the correlations (KM), the standard deviations (SD), and the means (ME), in that order. As the correlation matrix in the data file has been recorded with one row per line, the option SY on the KM command is essential; otherwise, the program would expect the correlations as one long line. The file EX101.LAB will be rewound after it has been read, so that the same labels may be read again for the next group. Note that the rewind (RE) option only applies to the file EX101.LAB, not to the file EX101.DAT.

The next five commands specify the model for the first group. The model includes the parameter matrices τ_x, κ, Λ_x, Φ, and Θ_δ. LX, PH, and TD are

Table 10.3
Nine Psychological Variables: Means and Standard Deviations

	Standard Deviations				*Means*			
Group	1	2	3	4	1	2	3	4
Visual Perception	1.06	0.96	0.95	1.03	4.20	4.30	4.29	4.21
Cubes	1.20	0.86	1.03	0.86	5.25	5.03	5.32	5.33
Paper Form Board	1.02	0.99	0.92	1.06	4.96	5.06	5.02	5.09
General Information	1.03	0.96	0.99	1.01	2.98	3.41	3.72	4.15
Sentence Completion	1.08	1.06	0.96	0.91	3.20	3.38	3.78	3.88
Word Classification	0.99	1.01	0.95	1.05	4.45	4.76	5.17	5.59
Figure Recognition	1.17	1.01	0.81	0.98	13.42	13.62	13.70	13.72
Object-Number	1.00	1.10	0.83	1.04	1.74	2.14	1.30	1.78
Number-Figure	1.04	1.00	0.88	1.07	2.10	2.16	1.87	2.44

default, TX is declared free for the first group and is declared invariant for the other groups, and KA is specified as fixed for the first group and free for the others. The OU command for the first group requests t-values (TV), standardized solution (SS), and 2 decimals (ND=2). Note that this information is automatically carried onto the OU command for groups 2–4. Similarly, the information on the MO command for the second group is automatically carried onto the MO commands for groups 3 and 4. For the same reason, only the NO keyword needs to be specified on the DA command for groups 2, 3, and 4. The general rule is that options and keywords that are the same as in the previous group need not be specified.

```
Holzinger-Swineford Data : LX and TD invariant : PASTEUR Low
DA NI=9 NG=4 NO=77
KM SY FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT
LA FI=EX101.LAB Rewind
MO NX=9 NK=3 TX=FR KA=FI
LK
Space Verbal Memory
PA LX
3(1 0 0) 3(0 1 0) 3(0 0 1)
FI LX 1 1 LX 4 2 LX 7 3
VA 1 LX 1 1 LX 4 2 LX 7 3
OU TV SS ND=2
```

```
Holzinger-Swineford Data : LX and TD invariant : PASTEUR High
DA NO=79
KM SY FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT
LA FI=EX101.LAB Rewind
MO LX=IN TX=IN KA=FR TD=IN
LK
Space Verbal Memory
OU
Holzinger-Swineford Data : LX and TD invariant : GRANT-WHITE Low
DA NO=74
KM SY FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT
LA FI=EX101.LAB Rewind
MO
LK
Space Verbal Memory
OU
Holzinger-Swineford Data : LX and TD invariant : GRANT-WHITE High
DA NO=71
KM SY FI=EX101.DAT;SD FI=EX101.DAT;ME FI=EX101.DAT
LA FI=EX101.LAB Rewind
MO
LK
Space Verbal Memory
OU
```

The maximum likelihood solution is shown in Table 10.4.

Example 10.2: Head start summer program

Source: Magidson (1977), Sörbom (1981).

Sörbom used data on 303 white children from the Head Start summer program, consisting of a Head Start sample (N = 148) and a matched Control sample (N = 155). The correlations, standard deviations, and means are given in Table 10.5. The children were matched on sex and kindergarten attendance but no attempt had been made to match on social status variables. The variables used in Sörbom's reanalysis were:

x_1	=	*Mother's education*	x_2	=	*Father's education*
x_3	=	*Father's occupation*	x_4	=	*Family income*
y_1	=	*Score on the Metropolitan Readiness Test*			
y_2	=	*Score on the Illinois Test of Psycholinguistic Abilities*			